3D Frame Element (12×12) — Derivation, Equivalent Nodal Loads, and Validation

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Scope. This document provides a derivation of the standard Euler-Bernoulli 3D frame element stiffness matrix (12×12) with 6 DOF per node, a consistent (equivalent) nodal load derivation for common distributed and point loads, and a set of validation benchmarks.

Assumptions. Small strains/displacements, linear elastic material behavior, prismatic member properties within an element, Euler-Bernoulli bending (shear deformation neglected), Saint-Venant torsion, and no warping DOF.

Contents

1. Degrees of Freedom, Sign Conventions, and Local Axes

Each frame element connects node i (end 1) to node j (end 2) and is defined in a right-handed local coordinate system (x, y, z) with x along the element from i to j. At each node there are 6 degrees of freedom (DOF):

```
Node DOF order (per node):  [ u  v  w  rx  ry  rz ]
u  = translation along local x (axial)
v  = translation along local y
w  = translation along local z
rx = rotation about local x (torsion)
ry = rotation about local y (bending in x-z plane)
rz = rotation about local z (bending in x-y plane)
```

Element displacement vector in local coordinates:

```
d_local = [u1 v1 w1 rx1 ry1 rz1  u2 v2 w2 rx2 ry2 rz2]^T
```

Internal force resultants at each end are collected as:

```
f_local = [Fx1 Fy1 Fz1 Mx1 My1 Mz1  Fx2 Fy2 Fz2 Mx2 My2 Mz2]^T
```

Positive shear forces and bending moments follow the usual beam sign convention consistent with the above DOF ordering.

2. Element Kinematics and Energy Decomposition

For an Euler-Bernoulli prismatic member, the strain energy decomposes into independent contributions from axial, torsion, and two orthogonal bending directions (about y and about z). This separability is what allows the 12×12 stiffness matrix to be assembled from four standard sub-problems.

Axial (bar) contribution:

```
U_axial = ∫_0^L (1/2) * (EA) * (du/dx)^2 dx
```

Torsion contribution:

```
U_torsion = ∫_0^L (1/2) * (GJ) * (drx/dx)^2 dx
```

Bending contributions (Euler-Bernoulli):

```
U_bend_z (v, rz) = ∫_0^L (1/2) * (EIz) * (d^2 v/dx^2)^2 dx
U_bend_y (w, ry) = ∫_0^L (1/2) * (EIy) * (d^2 w/dx^2)^2 dx
```

Total strain energy: U = U_axial + U_torsion + U_bend_y + U_bend_z. The element stiffness follows from the quadratic form U = (1/2) d_local^T K_local d_local.

3. Derivation of the Local 12×12 Frame Element Stiffness Matrix

Define element length L, axial rigidity EA, torsional rigidity GJ, and bending rigidities EIy and EIz (about local y and z respectively).

3.1 Axial and torsion submatrices

Using linear interpolation for u(x) and rx(x), the standard 2-node bar and torsion stiffness are:

```
k_axial = (EA/L) * [[ 1, -1],
                    [ -1,  1]]
k_tors  = (GJ/L) * [[ 1, -1],
                    [ -1,  1]]
```

These map directly into the u and rx DOF locations of the 12×12 matrix.

3.2 Bending submatrices (Hermite cubic interpolation)

For bending about z (v, rz) and about y (w, ry), use Hermite cubic shape functions. The resulting 4×4 Euler-Bernoulli beam stiffness in a single bending plane is:

```
k_bend(EI) = (EI/L^3) *
  [[ 12,   6L,  -12,   6L],
   [ 6L, 4L^2,  -6L, 2L^2],
   [-12,  -6L,   12,  -6L],
   [ 6L, 2L^2,  -6L, 4L^2]]
```

For bending about z: use EI = EIz with DOF order [v1, rz1, v2, rz2]. For bending about y: use EI = EIy with DOF order [w1, ry1, w2, ry2].

3.3 Full 12×12 local stiffness assembly

With DOF order [u v w rx ry rz] at each node, K_local is block-partitioned as:

```
K_local = [[K11, K12],
           [K21, K22]]
where each Kpq is 6×6 and K21 = K12^T, K22 = K11 (for prismatic members).
```

A convenient explicit form is shown below. Zero entries are omitted for readability; all unspecified terms are zero.

```
Let a = EA/L,  t = GJ/L,
    by = EIy/L^3,  bz = EIz/L^3.

K11 (6×6 at node 1 wrt node 1):
[ a,   0,    0,    0,     0,     0 ]
[ 0, 12bz,   0,    0,     0,   6L*bz]
[ 0,   0,  12by,   0,  -6L*by,  0 ]
[ 0,   0,    0,    t,     0,     0 ]
[ 0,   0, -6L*by,  0,  4L^2*by,  0 ]
[ 0, 6L*bz,  0,    0,     0,  4L^2*bz ]


K12 (6×6 coupling node 1 wrt node 2):
[ -a,    0,     0,    0,      0,      0 ]
[  0, -12bz,    0,    0,      0,   6L*bz ]
[  0,    0, -12by,    0,  -6L*by,   0 ]
[  0,    0,     0,   -t,      0,      0 ]
[  0,    0,  6L*by,   0,   2L^2*by,  0 ]
[  0, -6L*bz,   0,    0,      0,   2L^2*bz ]
```

Then K21 = K12^T and K22 equals K11 except for sign changes in the off-diagonal bending coupling terms consistent with the standard 4×4 beam stiffness in each plane. If you prefer an explicit 12×12 listing, see Appendix A.

Appendix A — Explicit 12×12 Local Stiffness Matrix

The following explicit 12×12 matrix uses the same abbreviations a, t, by, bz above. Rows/cols follow d_local = [u1 v1 w1 rx1 ry1 rz1 u2 v2 w2 rx2 ry2 rz2]^T.

```
K_local =
[  a,    0,      0,     0,        0,        0,   -a,     0,      0,     0,        0,        0 ]
[  0,  12bz,     0,     0,        0,     6L*bz,  0,   -12bz,     0,     0,        0,     6L*bz ]
[  0,    0,   12by,     0,    -6L*by,     0,     0,     0,   -12by,     0,    -6L*by,     0 ]
[  0,    0,      0,     t,        0,        0,   0,     0,      0,    -t,        0,        0 ]
[  0,    0,  -6L*by,    0,   4L^2*by,     0,     0,     0,   6L*by,     0,   2L^2*by,     0 ]
[  0,  6L*bz,    0,     0,        0,   4L^2*bz,  0,  -6L*bz,    0,     0,        0,   2L^2*bz ]
[ -a,    0,      0,     0,        0,        0,    a,     0,      0,     0,        0,        0 ]
[  0, -12bz,     0,     0,        0,   -6L*bz,  0,    12bz,     0,     0,        0,   -6L*bz ]
[  0,    0,  -12by,     0,     6L*by,     0,     0,     0,    12by,     0,     6L*by,     0 ]
[  0,    0,      0,    -t,        0,        0,   0,     0,      0,     t,        0,        0 ]
[  0,    0,  -6L*by,    0,   2L^2*by,     0,     0,     0,   6L*by,     0,   4L^2*by,     0 ]
[  0,  6L*bz,    0,     0,        0,   2L^2*bz,  0,  -6L*bz,    0,     0,        0,   4L^2*bz ]
```

4. Local-to-Global Transformation

Let R be the 3×3 direction cosine matrix mapping local axes to global axes. For a member with global end coordinates Xi, Xj, define the unit vector ex = (Xj - Xi)/L. Choose local ey, ez to complete a right-handed triad (e.g., using a global reference vector and Gram-Schmidt). Then:

```
R = [ex^T; ey^T; ez^T]  (rows are local basis vectors expressed in global coords)
```

The 12×12 transformation for a 3D frame element is block-diagonal in the sense that translations and rotations transform with the same 3×3 R:

```
T = diag(R, R, R, R)   (four 3×3 blocks)
d_local  = T * d_global
f_global = T^T * f_local
K_global_elem = T^T * K_local * T
```

Implementation note: if your global DOF order differs (e.g., [ux uy uz rx ry rz] per node), then T is assembled to match your exact DOF ordering.

5. Consistent (Equivalent) Nodal Load Vector Derivations

For an element load distribution p(x), the consistent nodal load vector is obtained from the virtual work statement:

```
δW_ext = ∫_0^L δu(x)^T p(x) dx  = δd_local^T f_eq
=> f_eq = ∫_0^L N(x)^T p(x) dx
```

where N(x) collects the appropriate shape functions for the DOF participating in the load (linear for axial/torsion, Hermite for bending). Below are closed-form results used most often in frame analysis.

5.1 Uniform distributed load in local y: qy (force/length)

Load acts along +y, produces shear forces Fy and bending moments about local z (Mz) via the (v, rz) bending plane.

```
Equivalent nodal forces/moments (local):
Fy1 =  qy*L/2
Fy2 =  qy*L/2
Mz1 =  qy*L^2/12
Mz2 = -qy*L^2/12
All other components = 0

f_eq_y = [0 Fy1 0 0 0 Mz1  0 Fy2 0 0 0 Mz2]^T
```

5.2 Uniform distributed load in local z: qz (force/length)

Load acts along +z, produces shear forces Fz and bending moments about local y (My) via the (w, ry) bending plane.

```
Equivalent nodal forces/moments (local):
Fz1 =  qz*L/2
Fz2 =  qz*L/2
My1 = -qz*L^2/12
My2 =  qz*L^2/12
All other components = 0

f_eq_z = [0 0 Fz1 0 My1 0  0 0 Fz2 0 My2 0]^T
```

5.3 Uniform axial load in local x: qx (force/length)

```
Fx1 = qx*L/2
Fx2 = qx*L/2
f_eq_x = [Fx1 0 0 0 0 0  Fx2 0 0 0 0 0]^T
```

5.4 Concentrated transverse load P at distance a from node 1 (0 <= a <= L)

For point loads, consistent nodal loads are obtained by evaluating the shape functions at x=a and distributing the load to the nodal DOF. For bending in the (v, rz) plane (load along local y), using Hermite functions:

```
Let xi = a/L.
Hermite shape functions for v(x):
N1 =  1 - 3xi^2 + 2xi^3
N2 =  L*(xi - 2xi^2 + xi^3)
N3 =  3xi^2 - 2xi^3
N4 =  L*(-xi^2 + xi^3)

For a point load P along +y at x=a:
Fy1 += P*N1
Mz1 += P*N2
Fy2 += P*N3
Mz2 += P*N4
(signs follow the same convention as Section 1)
```

Analogous expressions apply for a point load along local z acting in the (w, ry) plane by replacing (Fy, Mz) with (Fz, -My) consistent with Section 5.2.

5.5 Distributed load varying linearly (triangular) — closed form

For qy(x) = q1 + (q2 - q1) x/L, the consistent nodal load vector remains f_eq = ∫ N^T qy dx with Hermite N. XL4Sim can either (a) implement the closed form, or (b) evaluate the integral with 2-point Gauss quadrature (exact for up to cubic integrands). For robustness and extensibility, Gauss quadrature is recommended.

6. Validation and Benchmark Cases

These cases are intended for validation examples. All results assume Euler-Bernoulli behavior and small deflections.

6.1 Axial bar test

Single element, length L, area A, modulus E. Node 1 fixed in axial direction (u1=0). Apply axial force P at node 2 in +x.

```
Expected axial displacement at node 2:
u2 = P*L/(E*A)
Internal axial force is constant: N(x) = P
```

6.2 Torsion test

Single element, torsion constant J, shear modulus G. Fix rx1=0. Apply torque T at node 2 about +x.

```
Expected twist at node 2:
rx2 = T*L/(G*J)
Internal torque is constant: Mx(x) = T
```

6.3 Cantilever tip load (bending about z)

Cantilever along x, node 1 fixed, node 2 free. Apply transverse point load P at node 2 along +y.

```
Expected tip deflection and rotation (Euler-Bernoulli):
v2  = P*L^3/(3*E*Iz)
rz2 = P*L^2/(2*E*Iz)
Tip moment: Mz(0) = P*L
```

6.4 Simply supported beam with uniform load (bending about z)

Pinned-pinned support in the (v, rz) plane: v1=v2=0, rotations free. Uniform load qy along +y over full span.

```
Expected maximum deflection at midspan:
v_max = 5*qy*L^4/(384*E*Iz)
End moments should be ~0 for ideal pins (numerically small).
```

6.5 3D portal frame lateral stiffness (sanity check)

Two columns (height H) and a rigid beam (span B) forming a rectangular portal frame in the global X-Y plane. Fix both base nodes. Apply lateral force P at the top-left node in global +X. Compare to a fine-mesh reference (e.g., subdivide each member into 10+ elements) and verify convergence with mesh refinement.

This benchmark checks (a) transformation correctness, (b) assembly, and (c) bending coupling in global coordinates.

6.6 Energy check (general)

For any solved linear model: verify that external work equals strain energy at equilibrium:

```
U = (1/2) d^T K d
W = (1/2) d^T F
Expected: U ≈ W (within numerical tolerance)
```

7. Implementation Notes for XL4Sim

Recommended engineering checks to prevent common failure modes:

- Validate element length L > 0 and that the local axis construction does not degenerate (choose a robust reference vector).
- Check K_global symmetry after transformation/assembly (max|K - K^T| small).
- Detect unconstrained rigid body modes: rank(K_reduced) should match the number of free DOF; otherwise report a clear message.
- For moment releases/hinges, apply DOF condensation or modify the element stiffness in local coordinates before transformation.
- Use consistent units throughout (E, G, A, Iy, Iz, J, loads). Provide a units cheat sheet in the UI.

8. References

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